MAC-CPTM Situations Project

Situation 28: Adding Square Roots

(Includes material from Situation 32 Radicals)

Situation 28 Prepared at the University of Georgia
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Prompt

Mr. Fernandez is bothered by his ninth-grade algebra students' responses to a recent quiz on radicals, specifically those in response to a question about square roots in which students added $\sqrt{2}$ and $\sqrt{3}$ and got $\sqrt{5}$.

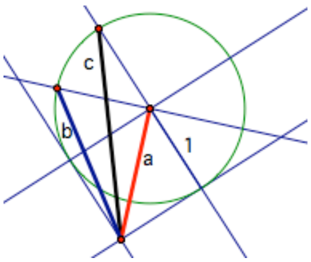
Commentary

The mathematical basis for determining the appropriateness of the students' work is that the sum of the square roots of two numbers is not, in general, equal to the square root of the sum of the two numbers. Establishing that a statement is not true can be accomplished in different ways, including finding a counterexample, indirect proof, and lack of correspondence of function graphs. The students' statement, $\sqrt{2}+\sqrt{3}=\sqrt{5}$, can be disproved using numeric, geometric, symbolic, and graphical representations. Connections are made to linear transformations from linear algebra and to multivariate functions and function composition.

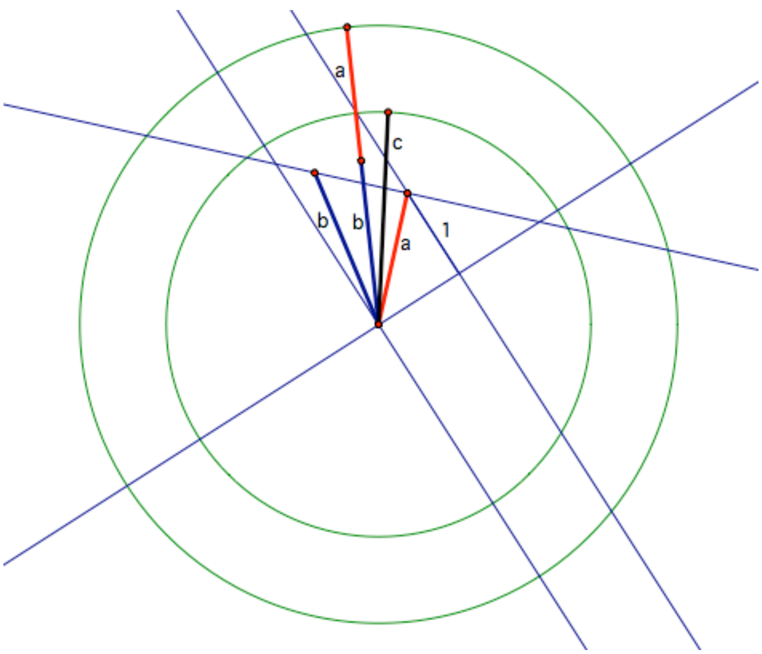
Mathematical Focus 1

Geometric constructions can be used to disprove $\sqrt{2}+\sqrt{3}=\sqrt{5}$.

Construct a circle with radius 1 and segments a, b, and c. Lines containing two of the diameters are perpendicular, and lines are drawn tangent to the circle at the intersection of these diameters with the circle. The third bisects two of the right angles formed by the perpendicular lines. This can be done using dynamic geometry software, as shown subsequently. In the figure, the Pythagorean theorem can be used to show that the lengths of segments a, b, and c must be $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$, respectively.



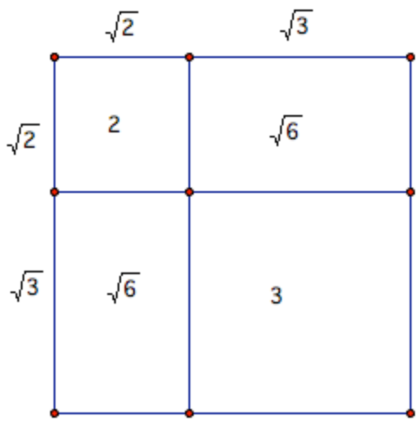
If $\sqrt{2}+\sqrt{3}=\sqrt{5}$ is a true statement, then the sum of the segment lengths a and b must be equal to the segment length c. As the following diagram illustrates, the sum of the segments lengths a and b is greater than the segment length c. Hence $\sqrt{2}+\sqrt{3}\neq\sqrt{5}$.

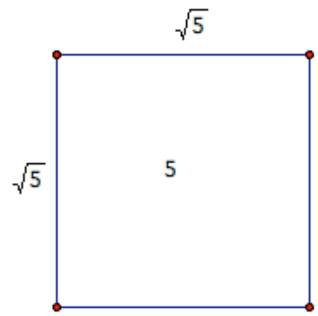


Mathematical Focus 2

A given statement can be shown to be false by supposing that it is true and showing that the supposition leads to a contradiction. This indirect proof uses the property "If $a = b$, then $a^2 = b^2$" and the fact that squares of numbers can be represented geometrically as areas of squares.

If $\sqrt{2}+\sqrt{3}=\sqrt{5}$ is a true statement, then $\sqrt{2}+\sqrt{3}=\sqrt{2+3}$ and $\left(\sqrt{2}+\sqrt{3}\right)^2=\left(\sqrt{2+3}\right)^2$ would also be true statements. Consider a square with side length $\sqrt{2}+\sqrt{3}$. The area of that square would be $\left(\sqrt{2}+\sqrt{3}\right)^2$. The square can be subdivided into four regions, each having area 2, $\sqrt{6}$, $\sqrt{6}$, and 3, as shown subsequently. Therefore, $\left(\sqrt{2}+\sqrt{3}\right)^2=2+\sqrt{6}+\sqrt{6}+3=5+2\sqrt{6}$. Now consider a square with side length $\sqrt{2+3}$. The area of that square would be $\left(\sqrt{2+3}\right)^2$, and $\left(\sqrt{2+3}\right)^2=\left(\sqrt{5}\right)^2=5$. Since $5+2\sqrt{6}\neq 5$, $\left(\sqrt{2}+\sqrt{3}\right)^2\neq\left(\sqrt{2+3}\right)^2$. Therefore, $\sqrt{2}+\sqrt{3}\neq\sqrt{5}$.





Mathematical Focus 3

A larger question than whether $\sqrt{2}+\sqrt{3}=\sqrt{5}$ is true is whether $\sqrt{a}+\sqrt{b}=\sqrt{a+b}$ is true for all $a,b\in\Re^+\cup\{0\}$. A statement such as $\sqrt{a}+\sqrt{b}=\sqrt{a+b}$ can be disproved by identifying values of a *and* b *that provide a counterexample.*

If $\sqrt{a}+\sqrt{b}=\sqrt{a+b}$ were true for all $a,b\in\Re^+\cup\{0\}$, then $\sqrt{2}+\sqrt{3}=\sqrt{2+3}$ would be true. But $\sqrt{2}+\sqrt{3}\doteq 1.414+1.732\doteq 3.15$ and $\sqrt{2+3}=\sqrt{5}\doteq 2.24$, so $\sqrt{2}+\sqrt{3}\neq\sqrt{5}$ and therefore $\sqrt{a}+\sqrt{b}\neq\sqrt{a+b}$.

AND

An expression such as $\sqrt{2}+\sqrt{3}$ can be thought of as a sum of function values, $f(2)+f(3)$, values of the square root function, $f(x)=\sqrt{x}$, $x\in\Re^+\cup\{0\}$. Some functions satisfy one of the conditions of the linearity property, namely, $f(a+b)=f(a)+f(b)$, and others do not. If a function f does not, in general, satisfy some property, it does not mean that there do not exist <u>some</u> values for which that property might hold.

If $\sqrt{2}+\sqrt{3}=\sqrt{2+3}$, then if $f(x)=\sqrt{x}$, $x\in\Re^+\cup\{0\}$, $f(2+3)$ must be equal to $f(2)+f(3)$. In general, when $f(x)=\sqrt{x}$ and $a,b\in\Re^+\cup\{0\}$, does $f(a+b)=f(a)+f(b)$?

In general, if a function in the real domain has the properties $f(a+b)=f(a)+f(b)$ and $f(ca)=c*f(a)$, then it is said to satisfy the linearity property. Linear functions of the form $f(x)=kx$ that represent a direct variation relationship between variables satisfy the linearity property. Thus $f(a+b)=f(a)+f(b)$ for such functions. This is true because for the linear function $f(x)=kx$, $x\in\Re$, for all $a,b\in\Re$, $f(a+b)=k(a+b)=ka+kb=f(a)+f(b)$.

To determine whether $f(a+b)=f(a)+f(b)$ holds for $f(x)=\sqrt{x}$, $x\in\Re^+\cup\{0\}$, one can attempt to find a counterexample. When $a=4$, $b=1$, $f(4+1)=\sqrt{4+1}=\sqrt{5}$ and $f(4)+f(1)=\sqrt{4}+\sqrt{1}=3$. Since $\sqrt{5}\neq 3$, we can conclude that $f(a+b)\neq f(a)+f(b)$ for $f(x)=\sqrt{x}$, $x\in\Re^+\cup\{0\}$. However, this only establishes that *f* does not (in general) satisfy the linearity property, it does not establish that for every two values of *a* and *b*, $f(a+b)\neq f(a)+f(b)$. For example, $\sqrt{0}+\sqrt{0}=\sqrt{0+0}$. So this linearity property argument does not necessarily establish that $\sqrt{2}+\sqrt{3}\neq\sqrt{2+3}$, it only establishes that the statement in question cannot be deemed true by using an argument that all such statements are true for this particular function. Hence, one can evaluate $\sqrt{2}+\sqrt{3}$ and $\sqrt{2+3}$, as done previously, to disprove $\sqrt{2}+\sqrt{3}=\sqrt{2+3}$.

Mathematical Focus 4
Expressions in two variables can be thought of as function values for multivariate functions. Function composition is not commutative.

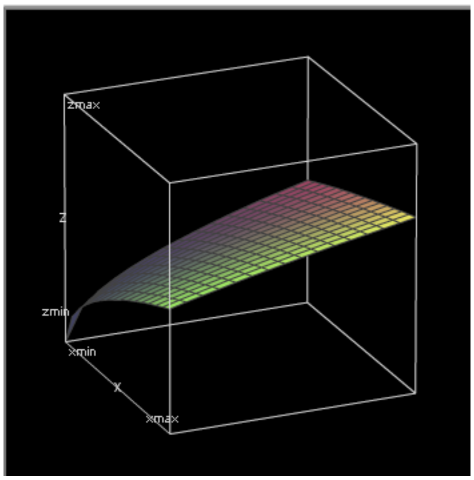
Consider $f(x,y)=\sqrt{x+y}$ with domain $x+y\in\Re^+\cup\{0\}$ such that $x,y\in\Re^+\cup\{0\}$. Another way to express $f(x,y)=\sqrt{x+y}$ is as the composition of two other functions, $g(x,y)=x+y$, with domain $x,y\in\Re$ and $h(x)=\sqrt{x}$ with domain $x\in\Re$. Since $f(x,y)$ is the composition of the functions represented by $g(x,y)=x+y$ and $h(x,y)=\sqrt{x}$, we can express $f(x,y)$ as $f(x,y)=h(g(x,y))=h(x+y)=\sqrt{x+y}$.

Now consider the function $s(x,y)=\sqrt{x}+\sqrt{y}$ with domain $x,y\in\Re^+\cup\{0\}$. We can express $s(x,y)$ as the composition of the functions represented by $g(x,y)=x+y$ and $h(x,y)=\sqrt{x}$, where $s(x,y)=g(h(x,y))=g\left(\sqrt{x},\sqrt{y}\right)=\sqrt{x}+\sqrt{y}$.

The composition $s(x,y) = g \circ h(x,y) = g(h(x,y))$ applies the square root first, then the sum; whereas the composition $f(x,y) = h \circ g(x,y) = h(g(x,y))$ applies the sum first, then the square root. If the composition operation on functions were commutative then the functions $f(x,y) = \sqrt{x+y}$ and $s(x,y) = \sqrt{x} + \sqrt{y}$ would be equal.

However, the graphs of $f(x,y) = \sqrt{x+y}$ and $s(x,y) = \sqrt{x} + \sqrt{y}$ are not identical. Therefore, the functions $f(x,y) = \sqrt{x+y}$ and $s(x,y) = \sqrt{x} + \sqrt{y}$ are not equal, and composition is not commutative.

$f(x,y) = \sqrt{x+y}$



$s(x,y) = \sqrt{x} + \sqrt{y}$